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ON CERTAIN FINITE DIMENSIONAL NUMERICAL RANGES AND NUMERICAL RA--ETC(U)
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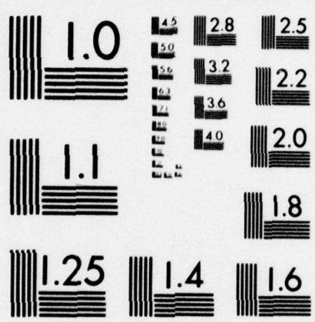

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LEVEL II

ON CERTAIN FINITE DIMENSIONAL NUMERICAL

RANGES AND NUMERICAL RADII*

by

Moshe Goldberg**

Department of Mathematics

University of California

Los Angeles, California 90024

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0. Introduction

The limited purpose of this paper is to discuss a few aspects of finite dimensional numerical ranges and numerical radii. There is no attempt to cover the entire field which has expanded in recent years almost beyond recognition. The aspects involved are merely those relating to the author's work in the past five years.

The paper consists of two parts: The first (Sections 1 - 3) discusses properties of the classical numerical range and the corresponding numerical radius; the second (Sections 4 - 6) is devoted to C-ranges and C-radii which are generalizations of the classical concepts.

Regrettably, we treat only a fraction of the finite dimensional theory. The omitted material includes topics concerning the classical range, as well as interesting generalizations due to Givens [19], Bauer [2], Marcus [38], Saunders and Schneider[49], and others.

The ranges and radii we treat here are associated with $n \times n$ matrices operating on $\underset{\sim}{C}^n$. Although the infinite dimensional aspects are neglected, many (but not all) of the mentioned results hold for linear operators on arbitrary Hilbert spaces as well. For the theory in general spaces the reader is referred to the thorough texts by Bonsall and Duncan [6, 7].

1. The Classical Numerical Range

The classical numerical range of a complex matrix $A \in \underset{\sim}{C}_{n \times n}$ (often called the field of values) is the compact set

$$W(A) = \{(Ax,x) : x \in \underset{\sim}{C}^n, \ |x| = 1\} .$$

Here (x,y) and $|x|$ are the standard inner product and norm defined by

$$(x,y) = x^*y \quad \text{and} \quad |x| = (x,x)^{1/2} .$$

The first to study $W(A)$ was Toeplitz [52] who showed that the boundary, $\partial W(A)$, is a convex curve. A year later Hausdorff [32] obtained the following celebrated result known as the Toeplitz-Hausdorff theorem.

THEOREM 1.1. The numerical range is a convex set in the complex plane.

Different proofs of the theorem can be found in [32, 55, 50, 39, 10, 30, 13, 31, 48, 9].

While for $A \in \mathbb{C}_{n\times n}$, $n \geq 3$, the explicit geometry of $W(A)$ is usually complicated (e.g. Murnaghan [40], Kippenhahn [36]), the 2×2 case is surprisingly simple. Murnaghan [40] (compare Donaghue [10]) has shown the following.

THEOREM 1.2. If

$$A = \begin{pmatrix} \lambda_1 & \alpha \\ 0 & \lambda_2 \end{pmatrix},$$

then $W(A)$ is the (possibly degenerate) elliptic disc with foci at λ_1, λ_2, and semi-minor axis $\frac{1}{2}|\alpha|$.

Since for any $A \in \mathbb{C}_{n\times n}$, $W(A)$ is invariant under unitary similarities of A, i.e.,

$$W(U^*AU) = W(A), \quad U \text{ unitary},$$

Theorem 1.2 treats in fact the general 2×2 case.

For larger matrices, one can obtain information on $W(A)$ by considering inclusion domains (e.g. [43, 33, 1]), or by approximating $W(A)$ numerically [35].

Easily verified properties of $W(A)$ are:

$$W(\alpha A) = \alpha W(A), \quad \alpha \in \underset{\sim}{C};$$
$$W(A + \alpha I) = W(A) + \alpha;$$
$$W(A + B) \subseteq W(A) + W(B);$$
$$\sigma(A) \equiv \{\text{spectrum } A\} \subseteq W(A).$$

Since $W(A)$ is convex it also follows that

$$W(A \oplus B) = \text{conv}\{W(A), W(B)\} \qquad (\text{conv for convex hull}), \tag{1.1}$$

and

$$P(A) \equiv \text{conv}\,\sigma(A) \subseteq W(A). \tag{1.2}$$

Review of these and other properties of $W(A)$ can be found for example in [44, 36, 51].

2. The Classical Numerical Radius

Associated with the numerical range is the numerical radius

$$r(A) \equiv \max\{|\zeta| : \zeta \in W(A)\}.$$

Following Ostrowski [42], we call a mapping $N : \underset{\sim}{C}_{n\times n} \to \underset{\sim}{R}$ a generalized matrix norm if for all $A, B \in \underset{\sim}{C}_{n\times n}$ and $\alpha \in \underset{\sim}{C}$,

$$N(A) \geq 0, \text{ with } N(A) > 0 \text{ for } A \neq 0;$$
$$N(\alpha A) = |\alpha| N(A);$$
$$N(A + B) \leq N(A) + N(B).$$

If in addition N is (sub-) multiplicative, namely,

$$N(AB) \leq N(A)N(B),$$

then N is called a matrix norm.

Having the above definitions one can easily prove,

THEOREM 2.1. *The numerical radius is a generalized matrix norm on* $\underset{\sim}{C}_{n\times n}$.

The numerical radius is not a matrix norm. For example, take

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2}, \quad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2}.$$

By Theorem 1.2 and (1.1), $r(A) = r(B) = 1/2$, $r(AB) = 1$. Consequently $r(AB) > r(A)r(B)$ and r is not multiplicative on $\underset{\sim}{C}_{n\times n}$, $n \geq 2$.

Brown and Shields (see [45]) have considered the 4×4 nilpotent matrix E, $E_{ij} = \delta_{i-1,j}$, showing that $r(E^2) = r(E^3) = \frac{1}{2}$ while $r(E) < 1$. Thus $r(E^3) > r(E^2)r(E)$. So in general the inequality $r(AB) \leq r(A)r(B)$ may fail even when A and B are powers of the same matrix.

What is true, however, is the following remarkable result by Berger [3].

THEOREM 2.2. *For any* $A \in \underset{\sim}{C}_{n\times n}$,

$$r(A^k) \leq r^k(A), \quad k = 1,2,3,\cdots.$$

An equivalent statement is that $r(A) \leq 1$ implies $r(A^k) \leq 1$, $k = 1,2,3,\cdots$.

The history of Theorem 2.2 is interesting, [4]: Lax and Wendroff [37] proved that $r(A) \leq 1$, $A \in \underset{\sim}{C}_{n\times n}$, implies $r(A^k) \subseteq \gamma(n)$ with $\gamma(n)$ depending on n alone and satisfying $\gamma(n) \to \infty$ as $n \to \infty$. It was Halmos who conjectured that $\gamma(n)$ can be replaced by 1. Two special cases followed: Bernau and Smithies [5] showed that $r(A) \leq 1$ implies $r(A^{2^k}) \leq 1$, $k = 1,2,3,\cdots$; and Brown [8] proved the desired result for 2×2 matrices. Berger [3] was the first to prove the general case, but his proof was never published. The first published proof is due to Pearcy [45].

Some properties of r follow immediately from what we know about $W(A)$. For example,

$$r(U^*AU) = r(A), \quad U \text{ unitary};$$
$$r(B) \le r(A), \quad B \text{ principal submatrix of } A;$$
$$r(A \oplus B) = \max\{r(A), r(B)\}.$$

It is also not hard to see that

$$\rho(A) \le r(A) \le \|A\|, \tag{2.1}$$

where

$$\rho(A) \equiv \max\{|\lambda| : \lambda \in \sigma(A)\}$$

and

$$\|A\| \equiv \max\{|Ax| : |x| = 1\}$$

are the spectral radius and spectral norm of A, respectively.

3. Convexity, Radiality, and Spectrality

In (1.2) and (2.1) we mentioned the relations

$$P(A) \subseteq W(A), \quad \rho(A) \le r(A) \le \|A\|.$$

It seems natural to ask when does equality hold.

Following Halmos [31], we say that A is convexoid if

$$P(A) = W(A);$$

radialoid if

$$\rho(A) = \|A\|;$$

<u>spectraloid</u> if

$$\rho(A) = r(A);$$

and <u>normaloid</u> if

$$r(A) = \|A\| .$$

Having defined four types of matrices, an interesting result of Wintner [56] shows that in fact we have only three:

THEOREM 3.1. A <u>is normaloid if and only if</u> A <u>is radialoid</u>.

Thus we may drop the term "normaloid" to avoid confusion with "normal" matrices.

The next three theorems characterize convexoids, radialoids and spectraloids.

THEOREM 3.2. A <u>is convexoid if and only if any of the following conditions holds</u>:

(a) (<u>Orland</u> [41]) $\|(A - \zeta I)^{-1}\| \le \{\text{distance}[\zeta, P(A)]\}^{-1}$, <u>for all</u> $\zeta \notin P(A)$.

(b) (<u>Furuta and Nakamoto</u> [15]) $A - \zeta I$ <u>is spectraloid for all</u> $\zeta \in \underset{\sim}{C}$.

(c) (<u>Furuta</u> [14]) $\operatorname{Re} P(e^{i\theta}A) = P(\operatorname{Re} e^{i\theta}A)$; $0 \le \theta \le 2\pi$.

(d) (<u>Johnson</u> [34]) A <u>is unitarily similar to</u> $B \oplus C$ <u>where</u> B <u>is normal and</u> $W(C) \subseteq W(B)$.

THEOREM 3.3. $A \in \underset{\sim}{C}_{n\times n}$ <u>is radialoid if and only if any of the following is satisfied</u>:

(a) (<u>e.g.</u> [47]) $\|A^k\| = \|A\|^k$, $k = 1,2,3,\cdots$.

(b) (<u>Pták</u> [46]) $\|A^n\| = \|A\|^n$.

(c) (Goldberg and Zwas [28]) $\|A^{\ell}\| = \|A\|^{\ell}$; ℓ any integer exceeding the degree of the minimal polynomial of A.

(d) (Goldberg and Zwas [26]) $\rho^2(A)I - A^*A$ is positive semi-definite.

(e) (Pták [47]) A is unitarily similar to $\|A\|(U \oplus B)$ where U is unitary and $\|B\| \leq 1$.

New proofs of Pták's interesting result in (b) were published by Flanders [12], Wimmer [54], and Pták [47]. The result in (c) -- merely a technical improvement of Pták's Theorem -- can be obtained by a careful inspection of [12].

THEOREM 3.4. A is spectraloid if and only if any of the following holds:

(a) (Furuta and Takeda [16]) $r(A^k) = r^k(A)$, $k = 1,2,3,\cdots$.

(b) (Goldberg, Tadmor and Zwas [24]) $r(A^{\ell}) = r^{\ell}(A)$, with ℓ as in Theorem 3.3 (c).

(c) (unpublished; follows from Theorem 1, [24]) A is similar to $r(A)(U \oplus B)$ where U is unitary and $r(B) \leq 1$.

Since the degree of the minimal polynomial of an $n \times n$ matrix never exceeds n, (b) implies that $A \in \mathbb{C}_{n\times n}$ is spectraloid if and only if $r(A^n) = r^n(A)$. Thus, we have an analogue of Theorem 3.2 (b).

It seems worth noting that for a general $A \in \mathbb{C}_{n\times n}$, $n \geq 2$, the equality

$$r(A^{n-1}) = r^{n-1}(A)$$

does not guarantee spectrality. For $n = 2$ the assertion is trivial;

for $n \geq 3$ it was shown, [25], that the $n \times n$ matrix

$$A = \mathrm{diag}(0,\sqrt{2},1,\cdots,1,\sqrt{2})E \ , \quad E_{ij} = \delta_{i-1,j} \ ,$$

satisfies $r(A^{n-1}) = r^{n-1}(A)$, but $r(A) > 0 = \rho(A)$.

The last result in this section describes the various inclusion relations between the classes of $n \times n$ convexoids, radialoids, spectraloids, and normal matrices, denoted by $\mathcal{C}_n$, $\mathcal{R}_n$, $\mathcal{S}_n$, and $\mathcal{N}_n$, respectively.

THEOREM 3.5. <u>Let</u> $\subset$ <u>denote proper inclusion. Then</u>

(a) $\mathcal{N}_2 = \mathcal{C}_2 = \mathcal{R}_2 = \mathcal{S}_2$.

(b) <u>For</u> $n = 3,4$ $\mathcal{N}_n = \mathcal{C}_n \subset \mathcal{R}_n \subset \mathcal{S}_n$.

(c) $\mathcal{N}_5 \subset \mathcal{C}_5 \subset \mathcal{R}_5 \subset \mathcal{S}_5$.

(d) <u>For</u> $n \geq 6$ $\mathcal{N}_n \subset \mathcal{C}_n \subset \mathcal{S}_n$, $\mathcal{N}_n \subset \mathcal{R}_n \subset \mathcal{S}_n$,

<u>but</u> $\mathcal{C}_n \not\subseteq \mathcal{R}_n$, $\mathcal{R}_n \not\subseteq \mathcal{C}_n$.

The relations between $\mathcal{N}_n$ and $\mathcal{C}_n$ are due to Moyls and Marcus [39]. The rest is due to Goldberg and Zwas [27].

4. C-<u>Numerical Ranges</u>

Let A, C be $n \times n$ complex matrices. Goldberg and Straus [21] call the set

$$W_C(A) = \{\mathrm{tr}(CU^*AU) : U \ \ n \times n \ \text{unitary}\}$$

the C-<u>numerical range</u> of A.

Evidently,

$$W(A) = W_C(A) \quad \text{with} \quad C = \mathrm{diag}(1,0,\cdots,0) \ ;$$

thus $W(A)$ is a special case of $W_C(A)$.

The set $W_C(A)$ can be viewed as the range of the mapping $\varphi : \mathcal{U}(A) \to \underset{\sim}{C}$ where

$$\mathcal{U}(A) = \{U^*AU : U \text{ unitary}\} ,$$

and φ is the linear functional on $\underset{\sim}{C}_{n\times n}$ defined by $\varphi(X) = \mathrm{tr}(CX)$. That is, $W_C(A)$ gives all the information a single functional provides about the class $\mathcal{U}(A)$ of matrices unitarily similar to A. From this point of view, $W_C(A)$ is the ultimate generalization of $W_C(A)$.

We remark that the above observation suggests a natural extention of the definition of $W_C(A)$ to arbitrary Hilbert spaces: If T is a linear bounded operator on a Hilbert space $\underset{\sim}{H}$ and φ is a linear functional on $\underset{\sim}{H}$, we define the φ-<u>numerical range</u> of T to be the set

$$W_\varphi(T) = \{\varphi(U^*TU) : U \text{ unitary on } \underset{\sim}{H}\} .$$

Clearly, A and C play a symmetric role in the definition of $W_C(A)$, i.e.,

$$W_C(A) = W_A(C) .$$

Also, $W_C(A)$ is invariant under unitary similarities of A (or of C). In fact we have ([21]),

THEOREM 4.1. $W_C(A) = W_{C'}(A)$ <u>for all</u> $A \in \underset{\sim}{C}_{n\times n}$ <u>if and only if</u> C <u>and</u> C' <u>are unitarily similar</u>.

The remainder of this section is devoted to C-ranges with normal C. Since C is normal if and only if C is unitarily similar to a diagonal matrix, Theorem 4.1 implies that we may restrict attention to ranges of the form $W_{diag(\gamma_1,\cdots,\gamma_n)}(A)$. For convenience we set the vector $c=(\gamma_1,\cdots,\gamma_n)$, and write $W_c(A)$ instead of $W_{diag(\gamma_1,\cdots,\gamma_n)}(A)$. We call $W_c(A)$ the c-numerical range of A.

A short calculation shows that

$$W_c(A) \equiv W_{diag(\gamma_1,\cdots,\gamma_n)}(A) = \left\{ \sum_{j=1}^{n} \gamma_j(Ax_j,x_j) \;:\; \{x_j\}_{j=1}^{n} \in \Lambda_n \right\},$$

where Λ_n is the set of all *orthonormal* bases of $\mathbb{C}^n$. Thus, $W_c(A)$ depends on the scalars $\gamma_1,\cdots,\gamma_n$ rather than on the ordered n-tuple c.

We recall now Halmos' definition of the k-numerical range [31, §167], which after normalization becomes

$$W_k(A) = \{\tfrac{1}{k}\mathrm{tr}(PAP) : P \text{ orthogonal projections of rank } k\},\ (1 \le k \le n).$$

It is a simple matter to check that $W_k(A)$ can be written as

$$(4.1) \qquad W_k(A) = W_{c_k}(A)\ , \quad c_k = \tfrac{1}{k}(e_1 + \cdots + e_k)\ ,$$

where $\{e_j\}_{j=1}^{n}$ is the standard basis of $\mathbb{C}^n$. Thus, k-numerical ranges are special cases of c-ranges. (Compare Fan's range [51, §8.2].)

The main result concerning $W_k(A)$ is due to Berger [31, §167].

THEOREM 4.2. The k-numerical range is convex.

This result was generalized for c-ranges by Westwick [53]:

THEOREM 4.3. If $c \in \mathbb{R}^n$ then $W_c(A)$ is convex. If $c \in \mathbb{C}^n$, $n \ge 3$, then $W_c(A)$ may fail to be convex even for normal A.

Westwick's deep theorem avoids the 2-dimensional case $c = (\gamma_1,\gamma_2) \in \mathbb{C}^2$, $A \in \mathbb{C}_{2\times 2}$. In this case Goldberg and Straus [21] have shown that

$$W_c(A) = (\gamma_1 - \gamma_2)W(A) + \gamma_2 \operatorname{tr} A .$$

Applying Theorem 1.2, we find that $W_c(A)$ is an elliptic disc with foci at

$$\gamma_1\lambda_1 + \gamma_2\lambda_2 \quad \text{and} \quad \gamma_1\lambda_2 + \gamma_2\lambda_1 ,$$

where λ_1, λ_2, are the eigenvalues of A.

What is the shape of $W_c(A)$ for $c \in \mathbb{C}^n$, $n \geq 3$? E. G. Straus conjectures that $W_c(A)$ is star-shaped with respect to $\sum_j \gamma_j \operatorname{tr} A$ -- a point which always belongs to $W_c(A)$. This conjecture was verified for $n = 3$ (unpublished).

The shape of $W_c(A)$ for non-normal C is an open question.

5. Inclusion Relations for c-Ranges

An interesting result concerning k-numerical ranges was proved by Fillmore and Williams [11].

THEOREM 5.1. For any $A \in \mathbb{C}_{n\times n}$,

$$\{\frac{1}{n} \operatorname{tr} A\} = W_n(A) \subseteq W_{n-1}(A) \subseteq \cdots \subseteq W_1(A) = W(A) .$$

In this section we present some generalizations of Theorem 5.1, due to Goldberg and Straus [21].

DEFINITIONS. (a) For vectors $c, c' \in \mathbb{C}^n$, we write $c \prec c'$ if there exists a doubly stochastic matrix S such that $c = Sc'$.

(b) The vector c is obtained from c' by pinching if two components γ_i', γ_j' of c' are replaced by γ_i, γ_j with

$$\gamma_i = \alpha\gamma_i' + (1-\alpha)\gamma_j' , \quad \gamma_j = (1-\alpha)\gamma_i' + \alpha\gamma_j' , \quad 0 \leq \alpha \leq 1 ,$$

while all other components of c' remain unchanged.

(c) We write $c \ll c'$ if c is obtained from c' by a finite succession of pinchings.

Note that the relations $c < c'$ and $c \ll c'$ are invariant under permutations of the components $\{\gamma_1,\cdots,\gamma_n\}$, $\{\gamma_1',\cdots,\gamma_n'\}$.

A less trivial remark is that $c \ll c'$ implies $c < c'$ but not conversely.

Having the above preliminaries we state,

THEOREM 5.2. If $c, c' \in \underset{\sim}{C}^n$, then

(a) $c \ll c'$ implies

$$W_c(A) \subseteq W_{c'}(A) \quad \forall A \in \underset{\sim}{C}_{n\times n} .$$

(b) We have $c < c'$ if and only if

$$W_c(A) \subseteq \text{conv}\, W_{c'}(A) \quad \forall A \in \underset{\sim}{C}_{n\times n} .$$

By Theorem 4.2, $W_{c'}(A)$ may fail to be convex, so we may not replace $\text{conv}\, W_{c'}(A)$ by $W_{c'}(A)$.

Given vectors $c, c' \in \underset{\sim}{C}^n$, it is usually difficult to check whether $c < c'$ or $c \ll c'$. For real vectors, however, the situation simplifies considerably.

DEFINITION. A real vector $c = (\gamma_1,\cdots,\gamma_n)$ is called ordered if

$$(5.1) \qquad \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n .$$

THEOREM 5.3. If $c = (\gamma_j)$, $c = (\gamma_j')$ are ordered n-vectors, then each of the relations $c < c'$, $c \ll c'$, is equivalent to

$$\sum_{j=1}^{k} \gamma_j \leq \sum_{j=1}^{k} \gamma_j' \qquad j = 1,2,\cdots,n ,$$

$$\bigcap_{A \in \underset{\sim}{C}_{n \times n}} W_c(A) = \{0\} .$$

More results in this vein are found in [20, 21, 22].

Inclusion relations for general C-ranges are unknown to us.

6. C-Numerical Radii

As in the classical case, we associate with the C-numerical range the C-numerical radius of A,

$$r_C(A) \equiv \max\{|\zeta| : \zeta \in W_C(A)\} .$$

Naturally, if $W_C(A)$ is a c-range or a k-range, the corresponding radius is denoted by $r_c(A)$ or $r_k(A)$, respectively.

We begin by utilizing some of the inclusion relations obtained in the previous section.

Obviously, if $W_c(A) \subseteq W_{c'}(A)$ or even if $W_c(A) \subseteq \operatorname{conv} W_{c'}(A)$, then $r_c(A) \leq r_{c'}(A)$, although the converse may fail to hold. Thus, Theorem 5.2 (b) yields:

THEOREM 6.1. *If* $c, c' \in \underset{\sim}{C}^n$ *satisfy* $c \prec c'$ *then*

$$r_c(A) \leq r_{c'}(A) \qquad \forall A \in \underset{\sim}{C}_{n \times n} .$$

Since $\ll$ implies $\prec$, then this is all we obtain from Theorem 5.2. Similarly, from Theorem 5.1 and Corollary 5.6 we have,

COROLLARY 6.2. (a) *The* k-*numerical radii satisfy*

$$\frac{1}{n} |\operatorname{tr} A| = r_n(A) \leq \cdots \leq r_1(A) = r(A) \qquad \forall A \in \underset{\sim}{C}_{n \times n} .$$

(b) *For* $c = (\gamma_j) \in \underset{\sim}{R}^n$ *with* $\sum_j \gamma_j = \alpha$,

<u>with equality for</u> $k = n$ (<u>compare</u> [29]).

REMARK 5.4. (a) Since $c \prec c'$ and $c \ll c'$ are invariant under permutations of the γ_j and γ_j', Theorem 5.3 implies that for arbitrary vectors $c,c' \in \mathbb{R}^n$, the relations $c \prec c'$ and $c \ll c'$ are equivalent.

(b) By the same argument, given $c,c' \in \mathbb{R}^n$, one may first rearrange the γ_j,γ_j' to satisfy (5.1), then apply Theorem 5.3 to determine whether $c \prec c'$ or not.

Theorem 5.2 (a) and Remark 5.4 (a) immediately imply,

THEOREM 5.5. <u>If</u> $c,c' \in \mathbb{R}^n$, <u>then</u> $c \prec c'$ <u>if and only if</u>

$$W_c(A) \subseteq W_{c'}(A) \qquad \forall A \in \mathbb{C}_{n\times n} .$$

By Theorem 5.3, the vectors c_k, $1 \le k \le n$, in (4.2) satisfy

$$c_n \prec c_{n-1} \prec \cdots \prec c_1 .$$

Thus, Theorem 5.1 follows immediately from Theorem 5.5.

Finall , using Theorem 5.5, 5.3, and Remark 5.4 (b), it is not hard to obtain,

COROLLARY 5.6. (a) <u>If</u> $c = (\gamma_j) \in \mathbb{R}^n$ <u>with</u> $\sum_j \gamma_j = \alpha$, <u>then</u> $(\alpha/n,\cdots,\alpha/n) \prec c$, <u>and hence</u>

$$\{\frac{\alpha}{n}\text{tr } A\} \subseteq W_c(A) \qquad \forall A \in \mathbb{C}_{n\times n} .$$

(b) <u>If</u> $\gamma_j \ge 0$, <u>then</u> $c \prec (\alpha,0,\cdots,0)$; <u>thus</u>

$$W_c(A) \subseteq \alpha W(A) \qquad \forall A \in \mathbb{C}_{n\times n} .$$

(c) <u>If</u> $\alpha = 0$, <u>then</u>

$$\frac{|\alpha|}{n}\,|\mathrm{tr}\;A| \le r_C(A) \qquad \forall A \in \underset{\sim}{C}_{n\times n}\;;$$

<u>and if</u> $\gamma_j \ge 0$, <u>then</u>

$$r_C(A) \le \alpha r(A) \qquad \forall A \in \underset{\sim}{C}_{n\times n}\;.$$

In the remainder of the section we discuss results, due to Goldberg and Straus [23], concerning the norm properties of the mapping

$$r_C : \underset{\sim}{C}_{n\times n} \to \underset{\sim}{R}\;.$$

Since for $n = 1$ the situation is trivial, <u>we assume from now on that</u> $n \ge 2$.

It is easy to see that r_C is a semi-norm, that is, for any $A,B \in \underset{\sim}{C}_{n\times n}$ and $\alpha \in \underset{\sim}{C}$,

$$r_C(A) \ge 0\;;$$
$$r_C(\alpha A) = |\alpha|\, r_C(A)\;;$$
$$r_C(A + B) \le r_C(A) + r_C(B)\;.$$

The following result characterizes those C for which r_C is positive definite, i.e., r_C is a generalized matrix norm.

THEOREM 6.3. r_C <u>is a generalized matrix norm if and only if</u> C <u>is not a scalar matrix and</u> $\mathrm{tr}\;C \ne 0$.

In particular, since the classical radius is generated by $C = \mathrm{diag}(1,0,\cdots,0)$, Theorem 6.3 implies Theorem 2.1.

REMARK 6.4. For $c = (\gamma_j) \in \underset{\sim}{C}^n$ we have

$$r_c(A) = r_{\mathrm{diag}(\gamma_1,\cdots,\gamma_n)}(A)\;.$$

Thus, by the last theorem, r_C is a generalized matrix norm if and only if not all the γ_j are equal and $\sum_j \gamma_j \neq 0$.

The multiplicativity of r_C is a much more complicated question to which we have a theoretical answer:

THEOREM 6.5. <u>A generalized matrix norm</u> N <u>is a matrix norm if and only if</u>

$$\nu_N \equiv \max\{N(AB) : N(A) = N(B) = 1\} \leq 1 . \tag{6.1}$$

In practice, however, Theorem 6.5 offers limited help since in general, ν_N is not available.

A different approach to multiplicativity is the following. Given a generalized matrix norm N, and a constant $\nu > 0$, we immediately observe that

$$N_\nu \equiv \nu N$$

is a generalized matrix norm too. If the new norm N_ν is multiplicative we say that ν is a <u>multiplicativity factor</u> for N.

It is not hard to see that ν <u>is a multiplicativity factor for</u> N <u>if and only if</u> $\nu \geq \nu_N$ where ν_N is given in (6.1). But again, the difficulty in obtaining ν_N for C-numerical radii renders this result useless. The only case in which ν_N was found (not via (6.6) though) is the following.

THEOREM 6.6. <u>For the classical radius we have</u> $\nu_r = 4$; <u>i.e.,</u> νr <u>is a matrix norm if and only if</u> $\nu \geq 4$.

A more practical way to obtain multiplicativity factors is suggested by a theorem of Gastinel [18] (originally in [17]).

THEOREM 6.7. Let N be a generalized matrix norm, M a matrix norm, and $\eta \geq \xi > 0$ constants such that

$$\xi M(A) \leq N(A) \leq \eta M(A), \quad \forall A \in \underset{\sim}{C}_{n\times n}.$$

Then, any ν with $\nu \geq \eta/\xi^2$ is a multiplicativity factor for N, i.e., $N_\nu \equiv \nu N$ is a matrix norm.

Taking M to be the spectral norm, Gastinel's theorem was useful in obtaining multiplicativity factors for c-radii, $c \in \underset{\sim}{R}^n$.

THEOREM 6.8. (a) Let $c = (\gamma_j)$ be a real n-vector such that not all the γ_j are equal and $\sum_j \gamma_j \neq 0$. Denote

$$\alpha = \left|\sum_{j=1}^{n} \gamma_j\right|, \quad \beta = \sum_{j=1}^{n} |\gamma_j|, \quad \delta = \max_{i,j} |\gamma_i - \gamma_j|.$$

Then for any ν with

$$\nu \geq 4\beta\left(\frac{2\alpha + \delta}{\alpha\delta}\right)^2,$$

the numerical radius $\nu r_c \equiv r_{\nu c}$ is a matrix norm on $\underset{\sim}{C}_{n\times n}$.

(b) If the γ_j are of the same sign (≥ 0 or ≤ 0), then $\nu r_c \equiv r_{\nu c}$ is a matrix norm for any

$$\nu \geq \frac{16\alpha}{\delta^2}.$$

Note that for γ_j of the same sign,

$$4\beta\left(\frac{2\alpha + \delta}{\alpha\delta}\right)^2 \geq \frac{16\,\alpha}{\delta^2};$$

hence in this case, (b) is a stronger result than (a).

By Remark 6.4 we observe that Theorem 6.8 provides multiplicativity factors only for those $c \in \underset{\sim}{R}^n$ for which r_c is a generalized matrix norm. This is consistent with the fact that indefinite non-trivial semi-norms have no multiplicativity factors at all.

Finally, apply Theorem 6.8 (b) to the vectors c_k in (4.1), which generate the k-numerical radii r_k. We find that νr_k, $1 \leq k \leq n-1$, is a matrix norm on $\underset{\sim}{C}_{n\times n}$ if $\nu \geq 16k^2$. Thus, the smallest multiplicativity factor Theorem 6.8 provides for the classical radius $r \equiv r_1$ is $\nu = 16$, which is far off the least multiplicativity factor $\nu_r = 4$ given by Theorem 6.6. In this light it seems worthwhile to improve Theorem 6.8, as well as to obtain multiplicativity factors for C-radii with general C.

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This expository solicited paper consists of two parts: The first discusses properties of the classical numerical range and the corresponding numerical radius; the second is devoted to C-ranges and C-radii which are generalizations of the classical concepts. Although the ranges and radii treated here are associated with finite dimensional complex matrices, many of the mentioned results hold for linear operators on arbitrary Hilbert spaces as well.

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